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# A modification of Plonsey´s formula for biomagnetism Version with corrections

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**Abstract: A new formula is obtained here to relate the emf induced in a coil of a measuring system, with two fields: the so called impressed electric current density field (due to the action potentials in the membranes of the excitable cells located in the volume conductor of the biological tissues), and a certain stationary field, known as the magnetic lead field. The proposed formula modifies the one due to Robert Plonsey, published in 1972. After carrying out a review of the deduction of the original formula, its modification is deduced in the framework of electromagnetic theory and by analytical methods. The resultant modification emphasizes the weight assigned to the high frequencies in the induced emf. The deduction of the magnetic lead field is made here from a different approach. Some consequences of the modified formula are discussed and suggestions for further work are given. In Appendix A, the phasor field solutions of Maxwell equations corresponding to monochromatic oscillations in a general linear continuous medium are reviewed. Then, a general version of Lorentz reciprocity theorem for linear, nonhomogeneous, anisotropic, and lossy dispersive media is presented as a basis for deducing the modified formula proposed in this work. In Appendix B the restrictions that allow to apply the quasi-stationary approximation to electromagnetic theory in both, the quasi-electro-stationary and the magneto-quasi-stationary approaches are considered.**

**Key words: Plonsey´s formula of biomagnetism, reciprocity theorem, magnetic field leads, impressed electric currents, inhomogeneous and anisotropic volume conductor, quasi-stationary approximations.**

## Introduction

Organisms produce magnetic fields related with physiological electrical currents (such as currents associated with muscle contraction and the transfer of electrical signals through the nervous system) or with the magnetization of some body tissues.
Compared to the Earth's magnetic field, they are quite weak, but can be measured with instruments sensitive enough and under appropriate conditions to eliminate disturbances of external origin (Hobbie and Roth, 2007).

In the sixties of the last century the magnetic fields produced by the electrical activities of the heart and brain were measured. The magnetic fields were measured from the recording of the electromotive force induced by the fields, variable in time, in coils with millions of turns, locating the organism and the measurement system in isolated rooms (from the magnetic point of view) and averaging the signals (Baule and Mc Fee, 1965).
Starting in the following decade, much more sensitive instruments, based on superconducting quantum interference (SQUID), began to be developed and used. Combined with the possibilities offered by multichannel electronics, new signal processing algorithms and mathematical models of the current sources, SQUIDs made it possible to develop basic research and clinical diagnostic methods (Cheyne, 2006).
These methods are based on inferences about the distribution of electrical currents in biological tissues that can be obtained from measurements and interpretations of the magnetic fields produced by these currents.
Moreover, the electrical currents induced in biological tissues by variable external magnetic fields are used in the framework of therapeutic methods, to stimulate excitable tissue. The spatial distribution of electrical energy of these induced currents is in general

different from that which can be obtained by means of contact electrodes (Malmivuo and Plonsey, 1995; Malmivuo, 1996).

The interval of frequencies involved in tissue´s biomagnetism is mainly comprised between 0 and 10 kHz. Indeed, the maximum significant frequency should be 1 kHz, due to the rise time of the action potentials in the electrically active tissues.
So, given the lengths involved, all the fields can be treated as quasi-stationary (Plonsey and Hepner, 1967).
Under quasi-stationary conditions each field oscillates in phase at different points in the material medium (Landau and Lifshitz, 1960; Nikolski, 1982; Haus and Melcher, 1989). Each field can be expressed as a product of a scalar function of time and a vector function of position in space.

In general, the solution of the inverse problem of determining the physiological currents given a measured magnetic field is often based in assuming a spatial distribution of a finite set of time variable dipolar sources. The parameters of this mathematical model are adjusted to the experimental results of measurement, using suitable optimization and statistical algorithms in the framework of a sequential process that includes several steps of increasingly better forward field calculations, comparison with measured fields and improved parameters estimations (Cheyne, 2006).

However, Robert Plonsey (Plonsey, 1972) showed that under quasi-stationary conditions, it is possible to derive a theoretical formula that relates, from one side, the electromotive force that appears in the induction coil with, from the other side, two vector fields: the so called impressed electric current density field in the biological volume conductor and a certain stationary field, known as the “magnetic lead field”.

The impressed currents that appear in Plonsey’s formula are the primary currents through the excitable membranes associated with action potentials in myocardium and nerve tissues. This primary, active currents are continued by secondary, passive currents in the volume conductor of the tissues. These last currents are conduction and displacement currents, both distinct from the impressed currents.
The other field that appears in Plonsey’s formula, the magnetic lead field, is a stationary field closely related with the electric field induced in the volume conductor by an external varying magnetic field. This last magnetic field is produced in the coil by an electric current impressed to the coil windings.
The name “magnetic lead field” was introduced to stress the analogy with the “electric lead field” that allows better interpretation of electrophysiological measurements (such as electrocardiogram and electroencephalogram) and helps in the design of measurement systems comprising contact electrodes located on the body surface (Malmivuo, 1996 and 2006).

Before going on with our subject, let us explain the notation that will be used in this article. Here, as usual $t$ represents the instant of time, $\vec{r}$ is the position vector and $\cdot$ represents the scalar product of two vectors.
Either one or two subscripts are used.
Only one subscript is used to refer to a vector field or a scalar function defined in one of the two subsystems, either the tissues or the coil, and considered as independent of what happens in the other subsystem. For example, the field of impressed current densities

$\vec{J}_T^{\,i}(t,\vec{r})$ generated in the tissues (subscript T) by their physiological activity, or an electric current $i_c(t)$ imposed in the coil by a current source connected to its terminals.
Two subscripts are used to refer to a vector field or a scalar function that constitutes a response in the tissues or in the coil (first subscript) to a phenomenon that occurs in the coil or in the tissues (second subscript), respectively. For example the electromotive force $v_{cT}(t)$ induced in the receiving coil (first subscript c) due to the field of impressed current densities generated in the tissues by their physiological activity (second subscript T).
A superscript *m* is used to refer to the magnetic lead vector field.

Now for the purpose of writing the formula derived by Plonsey, let us represent by $R_T$ the region of physical space occupied by the considered biological tissues. The formula relates the electromotive force $v_{cT}(t)$ induced in the receiving coil with the field of impressed current densities $\vec{J}_T^{\,i}(t,\vec{r})$ and with a magnetic lead field $\vec{E}_{Tc}^{\,m}(\vec{r})$:

$$v_{cT}(t) = \int_{R_T} \vec{J}_T^{\,i}(t,\vec{r}) \cdot \vec{E}_{Tc}^{\,m}(\vec{r})\, dV \qquad (1)$$

In formula (1) the magnetic lead field $\vec{E}_{Tc}^{\,m}(\vec{r})$ is obtained by imposing a variable electric current $i_c(t)$ to the windings of the receiving coil, in such a way that the temporal variation of the magnetic field generated by the coil, induces an electric field $\vec{E}_{Tc}(t,\vec{r})$ in the tissues. The magnetic lead field can be defined from this induced electric field (Plonsey, 1972).
However, in relation to formula (1) a difficulty arises if the impressed electric current density field in the tissues is stationary: $\vec{J}_T^{\,i}(t,\vec{r}) = \vec{J}_T^{\,i}(\vec{r})$ . In this case, formula (1) could predict a non-zero, time independent induced voltage in the windings of the coil.
Nevertheless, this voltage should be zero because in this case the magnetic field is everywhere independent of time and cannot produce an induced electromotive force in the coil windings.

We will show here that, also under quasi-stationary conditions, it is possible to obtain the following modified formula relating the electromotive force $v_{cT}(t)$ induced in the receiving coil with the field of impressed current densities $\vec{J}_T^{\,i}(t,\vec{r})$ in the tissues and a magnetic lead field in the tissues, now represented by $\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})$:

$$v_{cT}(t) = \int_{R_T} \frac{\partial \vec{J}_T^{\,i}}{\partial t}(t,\vec{r}) \cdot \vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\, dV \qquad (2.1)$$

Formula (2.1) can be written as a derivative with respect to time:

$$v_{cT}(t) = \frac{d}{dt} \int_{R_T} \vec{J}_T^{\,i}(t,\vec{r}) \cdot \vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\, dV \qquad (2.2)$$

The difference in the notation for the magnetic lead field is due to the fact that $\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})$ in formulas (2.1) and (2.2) cannot the same as $\vec{E}_{Tc}^{\,m}(\vec{r})$ in formula (1), among other

reasons because the physical dimensions of both fields are different (in SI units $\left[\vec{E}^{m}_{Tc}(\vec{r})\right]=\frac{V}{A\times m}$ and $\left[\vec{\mathcal{E}}^{m}_{Tc}(\vec{r})\right]=\frac{V\times s}{A\times m}$).

In both formulas, (1) and (2.1), the integrand is the scalar product of the local value of a source vector field (either $\vec{J}^{i}_{T}(t,\vec{r})$ or $\frac{\partial}{\partial t}\vec{J}^{i}_{T}(t,\vec{r})$ ) with the local value of a magnetic vector lead field (either $\vec{E}^{m}_{Tc}(\vec{r})$ or $\vec{\mathcal{E}}^{m}_{Tc}(\vec{r})$).

Both in the case of the original formula (1) and in the case of formula (2.1), for a given magnitude of the vector source (either $\vec{J}^{i}_{T}$ or $\frac{\partial}{\partial t}\vec{J}^{i}_{T}$ ), the integrand is proportional to the magnitude of the corresponding magnetic lead field and to the cosine of the angle between these fields.

For a given angle between the local vector source and the local magnetic lead field, the local contribution to the induced voltage is proportional to the magnitude of the local magnetic lead field. If this last magnitude is zero, the local contribution of the source field to the induced voltage will be zero, with independence of the source's magnitude and the angle between the vector source and the magnetic lead field.

If in a certain place the source vector is orthogonal to the magnetic lead vector, the contribution of the local source to the voltage in the coil will be zero, and if both vectors have the same direction the contribution will be positive and maximum.

These observations justify calling the magnetic lead field "field of sensitivities" of the magnetic measuring system. Figure 1 shows a qualitative sketch of the magnetic lead field lines produced by a single turn coil in a spherical isotropic and homogeneous volume conductor.

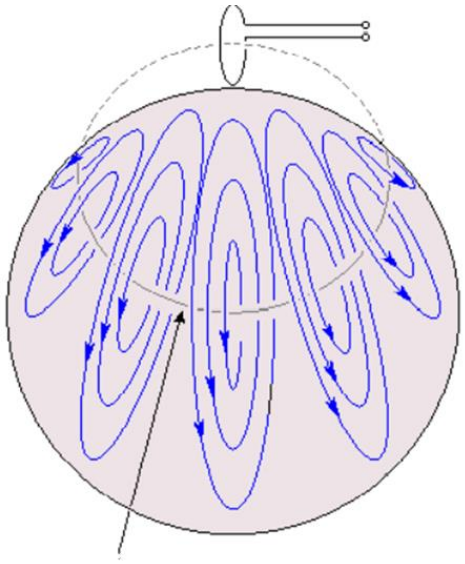

Figure 1 A energized single turn coil is located as suggested in the figure. The electric current that flows in the coil produce in a spherical, linear, isotropic, and homogeneous volume conductor the magnetic lead field line (the field lines of the induced electric field). The arrow points to the zero-sensitivity curve, at which points the magnitude of induced electric field is zero. The magnetic lead field lines are sketched as nested sets of curves, concentric with the line of zero-sensitivity. The magnitude of the sensitivity field increases with the distance from the zero- sensitivity curve (modified from Malmivuo and Plonsey, 1995).

The proposed modification of the Plonsey formula of biomagnetism (as the original one) can be applied, under quasi-stationary conditions, to any electrical current distribution in biological tissues and to any magnetic vector lead field.
As consequence, they can be applied to mathematical models of impressed currents based on a finite set of dipoles, such as the ones that are often used to solve the inverse problem in biomagnetism, as already mentioned above.

However, the results obtained from each formula are, of course, not equivalent.
Assuming, to facilitate analysis of the meaning of the formulas, that the fields $\vec{E}_{Tc}^{\,m}(\vec{r})$ and $\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})$ are proportional, the difference between formula (1) and formula (2.1) is due to the difference in the vector source field related with the impressed current, imposed by the electrical activity of the excitable membranes in the tissues. The partial derivative relative to time that appears in the modified formula (2.1) assigns a greater weight to the contribution of the higher frequencies to the voltage induced in the coil windings, compared to what occurs in the original formula.

Due to its generality, this kind of formulas allow to analyze the spatial sensitivity of any magnetic measurement system, with the consequences that this information may have in the interpretation of experimental results, in clinical diagnosis and in the design of measurement equipment for magnetic fields of biological origin.

The purpose of this communication is:

(a) Carry out a review of the deduction of formula (1)
(b) Deduce formula (2.1) (and consequently its equivalent version (2.2)) after loosening some of the original restrictions assumed by Plonsey on the electric currents imposed in the coil, on the electric field induced in the tissues by those currents, and on the electric properties of the tissue´s volume conductor.
(c) Give a sound electromagnetic foundation to the stationary field $\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})$ that appears in both formulas (2.1) and (2.2), discussing some of the implications of the greater weight assigned to the higher frequencies by these formulas compared to what happens according to the original Plonsey's formula (1) for biomagnetism.

All this is done in the framework of electromagnetic theory and by analytical methods.
For ease of reading, in Appendix A we briefly review Maxwell's equations for a continuous medium, putting emphasis in their vector phasor field solutions corresponding to monochromatic oscillations. Then we review a general version of Lorentz reciprocity theorem for linear, nonhomogeneous, anisotropic, and lossy dispersive media, as a basis for deducing the modified formula proposed in this work.

In Appendix B, also for ease of reading, we review the restrictions that allow us to apply the quasi-stationary approximation to electromagnetic theory.
The so-called electro-quasi-stationary (EQS) approximation is sketched. A general version of the EQS that includes capacitive effects (displacement currents) in the bulk of the volume conductor is reviewed first. Then we consider a more restricted version of the EQS, that includes capacitive effects only at the level of the biological membranes but not in the bulk of volume conductor.

Then the focus is shifted to the so-called magneto-quasi-stationary approximation (MQS) which is necessary to determine the stationary field $\vec{\mathcal{E}}^{m}_{Tc}(\vec{r})$ that appears in the modified formulas (2.1) and (2.2). In this approximation displacement currents are neglected. A regular perturbation analysis is developed to estimate the relative magnitudes of the primary magnetic field (due to the impressed electric current in the coil) and the secondary magnetic field (due to the induced electric currents in the tissues), in a nonhomogeneous and anisotropic volume conductor. When the external magnetic field is dominant and the skin depth is at least an order of magnitude greater that the size of the volume conductor, it is shown that the quasi-stationary approximation can be applied to both, the scalar potential and the vector potential of the electromagnetic field in biological tissues. As consequence, the introduction of $\vec{\mathcal{E}}^{m}_{Tc}(\vec{r})$ in the modified formula is justified.

## Plonsey's deduction revisited.

Plonsey obtained his formula starting from a suitable version of the Lorentz reciprocity theorem of electromagnetic theory. He deduced this version ab-initio, in the same paper, starting from Maxwell equations applied to a linear, conductive, isotropic, homogeneous, and nondispersive medium (Plonsey, 1972).

With the notation introduced in the present communication, if one region of sources is the coil region $R_c$ and the other region of sources is the region $R_T$ occupied by the volume conductor, the Carson reciprocity gives (the development of Carson reciprocity from Lorentz reciprocity can be found in Appendix A to the present communication):

$$\int_{R_c} \dot{\vec{J}}^{i}_{c}(\omega;\vec{r})\cdot\dot{\vec{E}}_{cT}(\omega;\vec{r})\,dV = \int_{R_T} \dot{\vec{J}}^{i}_{T}(\omega;\vec{r})\cdot\dot{\vec{E}}_{Tc}(\omega;\vec{r})\,dV \qquad (3)$$

As we show in the Appendix A, formula (3) can be applied to a more general medium, not only to a linear, homogeneous, isotropic volume conductor, with no dispersive behavior.

As explained in Appendix A, the phasor vector fields that appear in formula (3) are amplitudes of monochromatic oscillation with an angular frequency $\omega$.

In left hand side of the formula, the phasor vector amplitude $\dot{\vec{J}}^{i}_{c}(\omega;\vec{r})$ corresponds to an impressed harmonic electric current in the coil, while $\dot{\vec{E}}_{cT}(\omega;\vec{r})$ corresponds to the electric field induced in the coil windings due to a harmonic component of the magnetic field produced by the tissues.

In right hand side of the formula, the phasor vector amplitude $\dot{\vec{J}}^{i}_{T}(\omega;\vec{r})$ corresponds to the impressed harmonic electric current due to the electrical activity of the tissues, and $\dot{\vec{E}}_{Tc}(\omega;\vec{r})$ corresponds to the electric field induced in the tissues by a harmonic magnetic field produced by the coil.

If in the coil region $R_c$ the impressed currents flow through wires of negligible thickness, under quasi-stationary conditions the left-hand member of equation (3) can be approximated as follows (Plonsey, 1972; Nikolski, 1982; Kong, 1986):

$$\int_{R_c} \dot{\vec{J}}^{i}_{c}(\omega;\vec{r})\cdot\dot{\vec{E}}_{cT}(\omega;\vec{r})\,dV = I_c(\omega)\,V_{cT}(\omega) \qquad (4)$$

In equation (4) the amplitude of the oscillating impressed current to the coil is $I_c(\omega)$ and the amplitude of the oscillating induced voltage between the coil endings, due to the impressed electric current in the tissues is given by the line integral $V_{cT}(\omega)=\int_{\Gamma}\dot{\vec{E}}_{cT}(\omega;\vec{r})\cdot\vec{t}(s)\,ds$ where $\Gamma$ is the curve that represents the coil, $\vec{t}(s)$ is the tangent vector to the curve $\Gamma$ at the considered point, located by the intrinsic coordinate $s$ given by the arc length of the segment of curve comprised between one end of the coil and the point. The integrand $\dot{\vec{E}}_{cT}(\omega;\vec{r})\cdot\vec{t}(s)$ is the tangential component of the amplitude of the induced electric field in the coil.
From (3) and (4) we obtain:

$$I_c(\omega)\,V_{cT}(\omega)=\int_{R_T}\dot{\vec{J}}^{i}_{T}(\omega;\vec{r})\cdot\dot{\vec{E}}_{Tc}(\omega;\vec{r})\,dV \qquad (5)$$

Plonsey assumes:
(1) That $I_c(\omega)$ can be taken equal to 1 for all angular frequencies.

(2) That $\dot{\vec{E}}_{Tc}(\omega;\vec{r})$ is independent of the frequency in the frequency interval where it is not zero, and is equal to a field called magnetic lead field $\vec{E}^{m}_{Tc}(\vec{r})$.

These two assumptions imply: $$V_{cT}(\omega)=\int_{R_T}\dot{\vec{J}}^{i}_{T}(\omega;\vec{r})\cdot\vec{E}^{m}_{Tc}(\vec{r})\,dV \qquad (6)$$

Reconstituting the signal in the time domain, the voltage $v_{cT}(t)$ induced in the coil windings due to the electrical activity of the tissues is obtained:

$$v_{cT}(t)=\int_{-\infty}^{+\infty}V_{cT}(\omega)e^{j\omega t}d\omega=\int_{R_T}\left\{\int_{-\infty}^{+\infty}\dot{\vec{J}}^{i}_{T}(\omega;\vec{r})e^{j\omega t}d\omega\right\}\cdot\vec{E}^{m}_{Tc}(\vec{r})\,dV=\int_{R_T}J^{i}{}_{T}(t,\vec{r})\cdot\vec{E}^{m}_{Tc}(\vec{r})\,dV$$

In this way formula (1) is substantiated: $$v_{cT}(t)=\int_{R_T}J^{i}{}_{T}(t,\vec{r})\cdot\vec{E}^{m}_{Tc}(\vec{r})\,dV$$

The first hypothesis made by Plonsey ($I_c(\omega)=1$ for all angular frequencies) is equivalent to the hypothesis that the current in the coil, in the time domain, is a Dirac delta.
In this case the assumption of almost stationary fields (that is, the possibility of not taking into account the propagation time of electromagnetic disturbances and the Kelvin effect) is difficult to justify, even for small dimensions of tens of cm that we could be considering.
The second hypothesis made by Plonsey (that $\dot{\vec{E}}_{Tc}(\omega;\vec{r})=\vec{E}^{m}_{Tc}(\vec{r})$ is independent of the angular frequency $\omega$ in the frequency interval of interest) was justified by him in the measured behavior of the SQUID response as a function of frequency, on the basis of a reciprocity principle connecting the receiving with the emitting behavior of an electromagnetic device.

## Deduction of a modified version of the Plonsey formula

### (1) Derivation of the modified formula from a factorization of the phasor electric vector field in the tissues

Now, we will see that from equation (5) it is possible to obtain formula (2.1) posed at the end of the introduction to the present communication, without having to make the hypotheses that Plonsey made, with the sole exception of the hypothesis of quasi-stationary fields for frequencies between 0 and 10 kHz, which we will maintain.

If we can show that $\dot{\vec{E}}_{Tc}(\omega;\vec{r}) = j\omega I_c(\omega)\vec{\mathcal{E}}^m_{Tc}(\vec{r})$ with $\vec{\mathcal{E}}^m_{Tc}(\vec{r})$ being a real vector field, from this last equality and from (5) we obtain:

$$V_{cT}(\omega) = \int_{R_T} j\omega \dot{\vec{J}}^i_T(\omega;\vec{r})\cdot\vec{\mathcal{E}}^m_{Tc}(\vec{r})\,dV \qquad (7)$$

Considering that $\int_{-\infty}^{+\infty} j\omega \dot{\vec{J}}^i_T(\omega;\vec{r})e^{j\omega t}d\omega = \frac{\partial}{\partial t}J^i{}_T(t,\vec{r})$, from equation (7) **we obtain formula (2.1)** as desired:

$$v_c(t) = \int_{-\infty}^{+\infty} V_{cT}(\omega)e^{j\omega t}d\omega = \int_{R_T}\left\{\int_{-\infty}^{+\infty} j\omega \dot{\vec{J}}^i_T(\omega;\vec{r})e^{j\omega t}d\omega\right\}\cdot\vec{\mathcal{E}}^m_{Tc}(\vec{r})dV = \int_{R_T}\frac{\partial}{\partial t}J^i{}_T(t,\vec{r})\cdot\vec{\mathcal{E}}^m_{Tc}(\vec{r})\,dV$$

In turn, from formula (2.1) it follows formula (2.2), which is more convenient for analysis.

So, let us show now that $\dot{\vec{E}}_{Tc}(\omega;\vec{r}) = j\omega I_c(\omega)\vec{\mathcal{E}}^m_{Tc}(\vec{r})$ **under quasi-stationary conditions**, for frequencies between 0 and 10 kHz.

This phasor amplitude $\dot{\vec{E}}_{Tc}(\omega;\vec{r})$ corresponds to the electric field induced in the volume conductor by the time-varying external (relative to the tissues) **primary magnetic field** produced by the coil.

This induced electric field produces eddy currents in the tissue, and these currents produce their own time-varying **secondary magnetic fields**, and so on.

The decomposition of $\dot{\vec{E}}_{Tc}(\omega;\vec{r})$ in a factor $j\omega I_c(\omega)$ that depends of the frequency only and a factor $\vec{\mathcal{E}}^m_{Tc}(\vec{r})$ that depends of the position in the volume conductor only, can be based on the so-called magnetic-quasi-stationary approximation (MQS).

### (2) The magnetic-quasi-stationary approximation to the electric field induced in the volume conductor

MQS is the form that the quasi-stationary approximation to electrodynamics takes when a primary external magnetic field dominates the processes in a volume conductor, in this case, the biological tissues. The conditions that allow to apply quasi-stationary approximation, are discussed in Appendix B to the present work.

In the MQS approximation the displacement current is neglected, so equation (A.1.1) of Appendix A is reduced to:

$$\nabla \times \vec{H} = \vec{J}^{i} + \vec{J}_{cond} \tag{8.1}$$

The constitutive relation that relates the magnetic induction $\vec{B}$ with the magnetic field $\vec{H}$ in the tissues in bulk, and in the space between the volume conductor and the coil, regarding the calculation of the magnetic lead field, can be very well approximated by $\vec{B} = \mu \vec{H}$ with the magnetic permeability $\mu$ taking values very near its value for vacuum $\mu_0$. So (A.1.2) and (A.1.3) are reduced to:

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{8.2}$$

$$\nabla \cdot \vec{H} = 0 \tag{8.3}$$

In the MQS approximation, in the equation (A.1. 4) (Gauss' law) the density of free charges is a leftover quantity (Haus and Melcher, 1989).
The density of free charges can be found from the left-hand member of Gauss' law, determining first the electric displacement field from the electric field by the corresponding constitutive relation, after having determined the electric field independently of Gauss' law, as will be shown below.

As explained in Appendix B, in addition to neglecting the displacement current, the quotient between a characteristic dimension of the volume conductor and the thickness of the skin effect (Kelvin effect) must be of an order of magnitude less than unity to have a situation of quasi-stationary fields in which all fields can be expressed as products of a function of time and a function of space.

Another requirement for quasi-stationarity is to have a purely conductive medium. The measured variations of the conductivity with frequency (dispersion phenomena) should be neglectable between 0 and 10 kHz.
The following figure 2 shows representative values that suggest neglect, in a first approximation, dispersive effects in the real part of the conductivity for frequencies in the above-mentioned interval (Foster, 2000; Gabriel and Peyman, 2019).

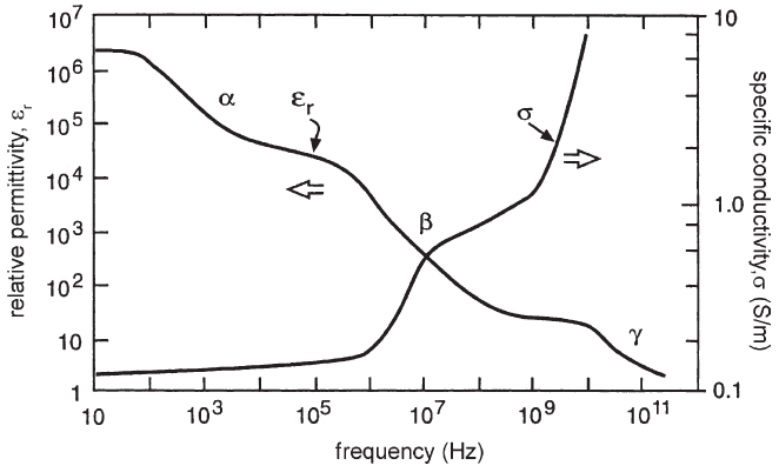


Figure 2 Representative values of relative permittivity (left ordinates) and conductivity (right ordinates) as functions of frequency for biological tissues.

So, we suppose that in this interval the components of the symmetric conductivity tensor of the biological tissues are real and functions of the position in the volume conductor: $\tilde{\sigma} = \tilde{\sigma}(\vec{r})$.

Then the conduction current in the tissues will be given by:

$$\vec{J}_{cond,T}(t,\vec{r}) = \tilde{\sigma}_T(\vec{r}) \cdot \vec{E}_T(t,\vec{r}) \tag{9}$$

When the coil is energized by a source of electric current and the tissues are passive:

$$\nabla \times \vec{H}_{Tc}(t,\vec{r}) = \tilde{\sigma}_T(\vec{r}) \cdot \vec{E}_{Tc}(t,\vec{r}) \tag{10}$$

Taking the divergence in both members of (10):

$$\nabla \cdot \left( \tilde{\sigma}_T(\vec{r}) \cdot \vec{E}_{Tc}(t,\vec{r}) \right) = 0 \tag{11}$$

From equation (A.1.6) the electric field in the tissues $\vec{E}_{Tc}(t,\vec{r})$ due to the electric current impressed in the coil can be decomposed using the scalar and the vector potentials:

$$\vec{E}_{Tc}(t,\vec{r}) = -\nabla \varphi_{Tc}(t,\vec{r}) - \frac{\partial \vec{A}_{Tc}}{\partial t}(t,\vec{r}) \tag{12}$$

In the MQS approximation the primary magnetic field produced by the coil is dominant. The magnetic permeability is everywhere (in the volume conductor and in the medium between the coil and the tissues) practically the same constant $\mu \cong \mu_0$.

Let us represent by $\vec{A}_c(t,\vec{r})$ the vector potential due to the induction coil in an unbounded medium of constant magnetic permeability $\mu \cong \mu_0$. This vector potential can be considered as known.

If the secondary magnetic field due to the induced electric currents in the tissues can be neglected, the vector potential can be calculated as if the volume conductor of the tissues were not present. Then $\vec{A}_{Tc}(t,\vec{r}) \cong \vec{A}_c(t,\vec{r})$. So, only the scalar potential $\varphi_{Tc}(t,\vec{r})$ remains to be found.

From (11) and (12):
$$\nabla \cdot \left( \tilde{\sigma}_T(\vec{r}) \cdot \nabla \varphi_{Tc}(t,\vec{r}) \right) = -\nabla \cdot \left( \tilde{\sigma}_T(\vec{r}) \cdot \frac{\partial}{\partial t} \vec{A}_c(t,\vec{r}) \right) \tag{13}$$

At the boundary $\partial R_T$ of the volume conductor:

$$\vec{n} \cdot \vec{J}_{cond,Tc}(t,\vec{r}) = 0 \tag{14}$$

Formula (14) for a volume conductor of tissues that is linear, isotropic, and homogeneous reduces to the cancellation of the normal component of the electric field at the boundary (Plonsey and Heppner, 1967).

For anisotropic and heterogeneous volume conductor, the derivation of formula (14) is discussed in Appendix B.

From (9), (12) and (14) we obtain a non-homogeneous boundary condition for the gradient of the scalar potential, for every time instant and every point located in $\partial R_T$:

$$\vec{n} \cdot \tilde{\sigma}_T(\vec{r}) \cdot \nabla \varphi_{Tc}(t,\vec{r}) = -\vec{n} \cdot \tilde{\sigma}_T(\vec{r}) \cdot \frac{\partial}{\partial t} \vec{A}_c(t,\vec{r}) \tag{15}$$

The hypothesis of quasi-stationary fields implies that the quasi-stationary vector potential of the primary magnetic field in the volume conductor of the tissues can be factorized in the product of a function of time and a function of the location in space.

If $i_c(t)$ is the impressed electric current in the coil:

$$\vec{A}_c(t,\vec{r}) = i_c(t)\,\vec{\Lambda}_c(\vec{r}) \qquad \vec{r} \in R_T \tag{16}$$

Taking into consideration the factorization of the vector potential and the quasi-stationary nature of the fields, as shown in Appendix B, the following factorization can be applied to find the scalar potential:

$$\varphi_{Tc}(t,\vec{r}) = \left(\frac{d}{dt} i_c(t)\right)\Phi_{Tc}(\vec{r}) \qquad \vec{r} \in R_T \tag{17}$$

From (12), (16) and (17) the following nonhomogeneous Poisson-like equation is obtained:

$$\nabla\cdot\left(\tilde{\sigma}_T(\vec{r})\cdot\nabla\Phi_{Tc}(\vec{r})\right) = -\nabla\cdot\left(\tilde{\sigma}_T(\vec{r})\cdot\vec{\Lambda}_c(\vec{r})\right) \quad \vec{r} \in R_T \tag{18}$$

From (15), (16) and (17) this non-homogeneous Neumann-like boundary condition results:

$$\vec{n}\cdot\left(\tilde{\sigma}_T(\vec{r})\cdot\nabla\Phi_{Tc}(\vec{r})\right) = -\vec{n}\cdot\left(\tilde{\sigma}_T(\vec{r})\cdot\vec{\Lambda}_c(\vec{r})\right) \qquad \vec{r} \in \partial R_T \tag{19}$$

From (18) and (19) we can determine $\Phi_{Tc}(\vec{r})$ univocally (assuming enough regularity in the given functions $\tilde{\sigma}_T(\vec{r})$ , $\vec{\Lambda}_c(\vec{r})$ and in the spatial domain (Zauderer, 2006)).

As we already assumed known $i_c(t)$ and $\vec{\Lambda}_{Tc}(\vec{r})$, the electric field in the tissues induced by the varying magnetic field due to the coil is now determined as desired:

$$\vec{E}_{Tc}(t,\vec{r}) = \left\{\frac{d}{dt} i_c(t)\right\}\left\{-\nabla\Phi_{Tc}(\vec{r}) - \vec{\Lambda}_c(\vec{r})\right\} \tag{20}$$

**(3) The phasor electric vector field and the magnetic lead field**

For a complex harmonic oscillating current $I_c(\omega)e^{j\omega t}$ , from (20) the following formula for the phasor electric vector field is derived:

$$\dot{\vec{E}}_{Tc}(\omega;\vec{r}) = j\omega I_c(\omega)\left\{-\nabla\Phi_{Tc}(\vec{r}) - \vec{\Lambda}_c(\vec{r})\right\} = j\omega I_c(\omega)\vec{\mathcal{E}}^m_{Tc}(\vec{r}) \tag{21}$$

Thus, we have justified the equation $\dot{\vec{E}}_{Tc}(\omega;\vec{r}) = j\omega I_c(\omega)\vec{\mathcal{E}}^m_{Tc}(\vec{r})$, deducing it from the magneto-quasi-stationary approximation.

Then the **magnetic lead field** is given by the formula:

$$\vec{\mathcal{E}}^m_{Tc}(\vec{r}) = -\nabla\Phi_{Tc}(\vec{r}) - \vec{\Lambda}_c(\vec{r}) \tag{22}$$

The quasi-static vector potential of the primary magnetic field in the volume conductor of the tissues, can be written:

$$\vec{A}_c(t,\vec{r}) = i_c(t)\,\vec{\Lambda}_c(\vec{r}) \qquad \vec{r} \in R_T \tag{23}$$

When the coil can be modelled by a smooth curve $L$ represented by the vector function $\vec{r}(s)$ of the arc length $s$, with unit tangent vector $\vec{t}(s)$, and if the coil is immersed in a linear, unbounded homogeneous and isotropic magnetic medium, it produces a vector field $\Lambda_c(\vec{r})$ in the region $R_T$ that can be described by the formula:

$$\vec{\Lambda}_c(\vec{r}) = \frac{\mu}{4\pi}\int_L \frac{\vec{t}(s)}{\|\vec{r}-\vec{r}(s)\|} ds \qquad \vec{r} \in R_T \qquad \vec{r}(s) \in L \qquad (24)$$

In this formula $\|\vec{r}-\vec{r}(s)\|$ is the norm that expresses the distance between a point $\vec{r}(s)$ of the coil and a point $\vec{r}$ belonging to the region occupied by the tissues.
Formula (24) is applicable only to a coil immersed in an homogeneous medium, but an expression for $\vec{\Lambda}_c(\vec{r})$ can be derived in more realistic contexts if necessary.

In highly simplified mathematical models, the functions $\Phi_{Tc}(\vec{r})$ and $\vec{\Lambda}_c(\vec{r})$ can be determined by analytical methods (Eaton, 1992).
With more realistic models, numerical methods like finite elements or finite differences, amongst others (Scorretti, Burais, Fabregue, et al, 2004; Lin, 2019) can be applied to find the components of the magnetic lead field.

## Discussion and conclusions

The mathematical model of the volume conductor considered in this communication is a monodomain model (Wolter and de Munck, 2007) that is assumed to be appliable to the complex structure of intracellular spaces, cellular membranes and extracellular spaces found in biological tissues. The resultant volume conductor is in general nonhomogeneous and anisotropic.
In the case of myocardium, the anisotropy ratios of the intracellular and extracellular spaces are different and a bidomain model describes better the electric conductive properties (Plonsey, 1989; Hobbie and Roth, 2007; Henriquez and Ying, 2009; Janssen, Potyagaylo, Dössel and Oostendorp, 2018).
However, the original 1972 Plonsey formula was derived in the framework of a monodomain model (although this terminology was not used in those times).
So, the modified formula proposed here is derived also in a monodomain framework, assuming a linear, non-homogeneous and anisotropic volume conductor.

Let us begin again with Plonsey´s original formula (1) that we repeat here for the reader's convenience: $v_{cT}(t) = \int_{R_T} \vec{J}_T^{\,i}(t,\vec{r}) \cdot \vec{E}_{Tc}^{\,m}(\vec{r})\, dV$

The detailed relation between $\vec{E}_{Tc}^{\,m}(\vec{r})$ and the electric field $\vec{E}_{Tc}(t,\vec{r})$ induced by the energized coil in the volume conductor of the tissues, was not studied in Plonsey's article. The author assumes that $\vec{E}_{Tc}^{\,m}(\vec{r}) = \dot{\vec{E}}_{Tc}(\omega;\vec{r})$ is in fact a certain stationary electric field and he immediately proceeds to deduce his formula for the induced emf $v_{cT}(t)$ in the coil, leaving open the problem of given a more detailed meaning to the magnetic lead field.
This subject was taken up by Malmivuo in his 1976 doctoral thesis, leading to the progressive development of a complete unified electric and magnetic lead field theory (Malmivuo, 2016).
To describe non-homogeneity and anisotropy, the approach that allowed the development of lead field theory is to work with a set of homogeneous (and if necessary anisotropic) interconnected conductive parts.
Within this conceptual framework, a theory of bio magnetic measurements was developed through a sequence of increasingly complex mathematical models, both in

terms of the sources of impressed currents, the volume conductor considered, and the measurement system (Malmivuo and Plonsey, 1995; Malmivuo, 2006).

Instead of decomposing the bulk conductor into a set of phases with discontinuities in the conductivity and dielectric permittivity through phase boundaries, in this communication we work with smooth tensorial fields of conductivity and dielectric permittivity to describe the electric properties of the biological tissues.
Only at the boundary between the volume conductor and its environment, a discontinuity is introduced in the conductivity and dielectric permittivity parameters of the constitutive equations.
The use of a single conductivity tensor field in the volume conductor of the tissues facilitates a theoretical approach like the one intended here.
The more or less abrupt variations that occur when passing from one type of biological tissue to an adjacent one can be described, with as much fidelity as may be necessary, by smooth tensor fields whose variation is fast enough at the interfacial regions.
Within this different framework that supplies the mathematical model of a single conductor, inhomogeneous and anisotropic (not divided as usual, in a set of homogeneous phases) it was possible to show that $\dot{\vec{E}}_{Tc}(\omega;\vec{r}) = j\omega I_c(\omega)\vec{\mathcal{E}}^{m}_{Tc}(\vec{r})$ instead of the assumption (unnecessary if what you are looking for is to estimate the induced emf in a coil) $\dot{\vec{E}}_{Tc}(\omega;\vec{r}) = \vec{E}^{m}_{Tc}(\vec{r})$ made by Plonsey in equation (5), that we repeat here for the reader's convenience: $I_c(\omega)V_{cT}(\omega) = \int_{R_T} \dot{\vec{J}}^{i}_{T}(\omega;\vec{r})\cdot\dot{\vec{E}}_{Tc}(\omega;\vec{r})\,dV$

The decomposition of the vector phasor amplitude $\dot{\vec{E}}_{Tc}(\omega;\vec{r})$ in a complex, frequency dependent scalar factor $j\omega I_c(\omega)$, and a real space dependent vector field $\vec{\mathcal{E}}^{m}_{Tc}(\vec{r})$, was based on the MQS approach to quasi-stationary fields.
Both factors in the decomposition are well defined, either because one factor is proportional to the amplitude of an alternating current injected into the coil, or because the other factor is a vector field defined in the volume conductor by the equation (22) in terms of the solution of the partial differential equation (18) with the condition (19) in the boundary of the volume conductor, assuming the dominance of the primary magnetic field due to the coil.

The equation that gives the vector phasor amplitude $\dot{\vec{E}}_{Tc}(\omega;\vec{r})$ can be generalized to more than one coil in the measuring system, and the analytical approach used in the present research can be complemented or substituted by a numerical approach.

The other relevant field, $\dot{\vec{J}}^{i}_{T}(\omega;\vec{r})$, that appears in the left-hand member of equation (5), when multiplied by $j\omega$ gives the vector phasor amplitude of the partial time derivative of an oscillating local impressed electric current. For the purposes of the present work, the field of impressed electric currents is assumed known.

Having already considered the two vector fields whose scalar product constitutes the integrand in formula (2.1), we can now focus on the modified formulas for the induced emf in the measuring coil.
To discuss the modified formulas, it is convenient to combine formulas (2.1) and (2.2):

$$v_{cT}(t)=\int_{R_T}\frac{\partial \vec{J}_T^{\,i}}{\partial t}(t,\vec{r})\cdot\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\,dV=\frac{d}{dt}\left(\int_{R_T}\vec{J}_T^{\,i}(t,\vec{r})\cdot\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\,dV\right) \qquad (25)$$

From the volume integral $\int_{R_T}\vec{J}_T^{\,i}(t,\vec{r})\cdot\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\,dV$ it follows that at the points of the tissues where the field of electric current density $\vec{J}_T^{\,i}(t,\vec{r})$ is orthogonal to the steady field $\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})$, the contribution to the integral is zero. If they are parallel or antiparallel, the contribution to the integral (respectively positive or negative) will be maximum in absolute value.

Now, all components of the formula $v_{cT}(t)=\frac{d}{dt}\int_{R_T}\vec{J}_T^{\,i}(t,\vec{r})\cdot\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\,dV$ for the induced voltage $v_{cT}(t)$ in a coil, due to variations of the magnetic field generated by the tissues, have been given meaning.

The impressed currents in the original and in the modified formulas are the primary currents through the excitable membranes associated with action potentials in biological tissues.

The calculation of the magnetic lead field $\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})$ for a linear, anisotropic and heterogeneous volume conductor under quasi-stationary conditions is described, assuming a known and smooth enough tensorial conductivity field $\tilde{\sigma}_T(\vec{r})$ in each point of the region occupied by the biological tissues.

In the original formula the source field is the field $\vec{J}_T^{\,i}(t,\vec{r})$ of impressed electric current densities.

In the first version (2.1) of the modified formula (the member located in the middle of the equalities (25)), the source field is the field of the partial derivatives relative to time of the impressed electric current densities.

In the second version (2.2) of the modified formula (the member located in the right end of the equalities (25)) appears the time derivative of $\int_{R_T}\vec{J}_T^{\,i}(t,\vec{r})\cdot\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\,dV$.

Because of this difference, the modified formula predicts that for the same local sensitivity of the measurement system, and a given angle between the local vector lead field and the source term, a higher contribution to the voltage induced in the coil is not due to a higher value of the magnitude of the local current density but to a higher value of the magnitude of its rate of variation with time in case of (2.1).

Equivalently, in case of formula (2.2), a higher contribution to the voltage induced in the coil is due to a faster variation in time of the volume integral $P_{Tc}(t)=\int_{R_T}\vec{J}_T^{\,i}(t,\vec{r})\cdot\vec{\mathcal{E}}_{Tc}^{\,m}(\vec{r})\,dV$.

Due to its generality and the possibility of computing magnetic lead fields with high precision and high spatial resolution (using numerical methods) this kind of formula facilitates analysis of the spatial sensitivity of a magnetic measurement system and the assessment of the contribution to the measured signal of the impressed electric currents due to the physiological activity of the tissues.

The present research is framed in a purely analytical theoretical approach, as was the article published by Plonsey (Plonsey, 1972).

A more in-depth study of the modification proposed in the present work, to the original Plonsey´s formula for biomagnetism should be carried out using digital simulation software.
The results of suitably designed numerical experiments could be of interest beyond the mere comparison of the predictions of one and the other formula.

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## Appendixes

### Appendix A: Lorentz reciprocity for linear, nonhomogeneous, anisotropic, and lossy dispersive media with localized sources of the fields

Lorentz reciprocity theorem is obtained for monochromatic oscillating sources and electromagnetic fields in a linear medium under certain restrictions (Landau and Lifshitz, 1960; Plonsey, 1972; Nikolski, 1982; Pappas, 1988). An interesting historical approach to reciprocity in electromagnetic theory can be found in the article by Paul Poincelot (Poincelot, 1961).

Let us consider the Maxwell's equations for a continuous medium (Kong, 1986; Haus and Melcher, 1989) without introducing constitutive equations:

$$\nabla \times \vec{H} = \vec{J}^{\,i} + \vec{J}_{cond} + \frac{\partial \vec{D}}{\partial t} \qquad \text{(A.1.1)} \qquad\qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(A.1.2)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{(A.1.3)} \qquad\qquad \nabla \cdot \vec{D} = \rho_{free} \qquad \text{(A.1.4)}$$

Here, as usual, $\vec{H}(t,\vec{r})$ is the magnetic field, $\vec{B}(t,\vec{r})$ the magnetic induction field, $\vec{E}(t,\vec{r})$ the electric field, $\vec{D}(t,\vec{r})$ the displacement field, $\vec{J}^{\,i}(t,\vec{r})$ the imposed (impressed) electric current density, $\vec{J}_{cond}(t,\vec{r})$ the conduction current density and $\rho_{free}(t,\vec{r})$ the density of free charges. Also, as usual $\nabla\times$ represents the curl operator and $\nabla\cdot$ the divergence operator. From (A.1.3): $\vec{B} = \nabla \times \vec{A}$ (A.1.5)
In (A.1.5), $\vec{A}(t,\vec{r})$ is the vector potential of the electromagnetic field.

From (A.1.5) and (A.1.2): $$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \qquad \text{(A.1.6)}$$

In (A.1.6), $\varphi(t,\vec{r})$ is the scalar potential of the electromagnetic field.
Taking the divergence in both members of (A.1.1) and considering (A.1.4), the following version of the conservation of electric charge is derived:

$$\nabla \cdot \vec{J}^{\,i} + \nabla \cdot \vec{J}_{cond} + \frac{\partial \rho_{free}}{\partial t} = 0 \qquad \text{(A.1.7)}$$

Now, let us consider the vector phasor field solutions of Maxwell's equations corresponding to monochromatic oscillations of angular frequency $\omega$ (Landau and Lifshitz, 1960; Nikolski, 1982; Pappas, 1988; Rothwell and Cloud, 2001).

We substitute the fields $\vec{E}(t,\vec{r})$, $\vec{D}(t,\vec{r})$, $\vec{H}(t,\vec{r})$ and $\vec{B}(t,\vec{r})$ by $e^{j\omega t}\dot{\vec{E}}(\omega;\vec{r})$, $e^{j\omega t}\dot{\vec{D}}(\omega;\vec{r})$, $e^{j\omega t}\dot{\vec{H}}(\omega;\vec{r})$ and $e^{j\omega t}\dot{\vec{B}}(\omega;\vec{r})$, the conduction current $\vec{J}_{cond}(t,\vec{r})$ by $e^{j\omega t}\dot{\vec{J}}_{cond}(\omega,\vec{r})$, the sources $\vec{J}^{i}(t,\vec{r})$ by $e^{j\omega t}\dot{\vec{J}}^{i}(\omega;\vec{r})$ and $\rho_{free}(t,\vec{r})$ by $e^{j\omega t}\dot{\rho}_{free}(\omega;\vec{r})$.

Here $\dot{\vec{E}}(\omega;\vec{r})$, $\dot{\vec{D}}(\omega;\vec{r})$, $\dot{\vec{H}}(\omega;\vec{r})$, $\dot{\vec{B}}(\omega;\vec{r})$, $\dot{\vec{J}}_{cond}(\omega,\vec{r})$, $\dot{\vec{J}}^{i}(\omega;\vec{r})$ and $\dot{\rho}_{free}(\omega;\vec{r})$ are phasor vectorial amplitudes (this is represented by the point that appears over each arrow) and, as usual in electric engineering, $j=\sqrt{-1}$.

Substituting these vector phasor field solutions in Maxwell equations, we obtain the following equations in the frequency domain:

$$\nabla\times\dot{\vec{H}}(\omega;\vec{r})=\dot{\vec{J}}^{i}(\omega;\vec{r})+\dot{\vec{J}}_{cond}(\omega;\vec{r})+j\omega\dot{\vec{D}}(\omega;\vec{r}) \qquad \text{(A.2.1)}$$

$$\nabla\times\dot{\vec{E}}(\omega;\vec{r})=-j\omega\dot{\vec{B}}(\omega;\vec{r}) \qquad \text{(A.2.2)}$$

$$\nabla\cdot\dot{\vec{B}}(\omega;\vec{r})=0 \qquad \text{(A.2.3)}$$

$$\nabla\cdot\dot{\vec{D}}(\omega;\vec{r})=\dot{\rho}_{free}(\omega;\vec{r}) \qquad \text{(A.2.4)}$$

The following constitutive relations describe a linear, anisotropic, non-homogeneous and lossy dispersive medium in the frequency domain (Post, 1962; Kong, 1986; Pappas, 1988):

$$\dot{\vec{B}}(\omega;\vec{r})=\tilde{\mu}(\omega;\vec{r})\cdot\dot{\vec{H}}(\omega;\vec{r}) \qquad \text{(A.3.1)}$$

$$\dot{\vec{D}}(\omega;\vec{r})=\tilde{\varepsilon}(\omega;\vec{r})\cdot\dot{\vec{E}}(\omega;\vec{r}) \qquad \text{(A.3.2)}$$

$$\dot{\vec{J}}_{cond}(\omega;\vec{r})=\tilde{\sigma}(\omega;\vec{r})\cdot\dot{\vec{E}}(\omega;\vec{r}) \qquad \text{(A.3.3)}$$

The components of the magnetic permeability tensor $\tilde{\mu}(\omega;\vec{r})$, the dielectric permittivity tensor $\tilde{\varepsilon}(\omega;\vec{r})$ and the electric conductivity tensor $\tilde{\sigma}(\omega;\vec{r})$ (all of them defined in the frequency domain) are in general complex scalar fields. From now on, we suppose that they are symmetric.

Let us consider in the physical space $R^3$ two regions $R_1$ and $R_2$, each one with an independent impressed vector phasor current field, $\dot{\vec{J}}_1^{i}(\omega;\vec{r})$ and $\dot{\vec{J}}_2^{i}(\omega;\vec{r})$ respectively. Each source produces its own electromagnetic fields and reaction currents, so we have:

$$\nabla\times\dot{\vec{H}}_1=\dot{\vec{J}}_1^{i}+\dot{\vec{J}}_{cond,1}+j\omega\dot{\vec{D}}_1 \qquad \text{(A.4.1)}$$

$$\nabla\times\dot{\vec{E}}_1=-j\omega\dot{\vec{B}}_1 \qquad \text{(A.4.2)}$$

$$\nabla\times\dot{\vec{H}}_2=\dot{\vec{J}}^{\,i}_2+\dot{\vec{J}}_{cond,2}+j\omega\dot{\vec{D}}_2 \qquad \text{(A.5.1)} \qquad\qquad \nabla\times\dot{\vec{E}}_2=-j\omega\dot{\vec{B}}_2 \qquad \text{(A.5.2)}$$

From these equations and following a standard procedure (Landau and Lifshitz, 1960; Nikolski, 1982; Pappas, 1988; Rothwell and Cloud, 2001) we obtain:

$$\begin{aligned}\nabla\cdot\left(\dot{\vec{E}}_1\times\dot{\vec{H}}_2-\dot{\vec{E}}_2\times\dot{\vec{H}}_1\right)&=\left(\dot{\vec{J}}^{\,i}_1\cdot\dot{\vec{E}}_2-\dot{\vec{J}}^{\,i}_2\cdot\dot{\vec{E}}_1\right)+\left(\dot{\vec{J}}_{cond,1}\cdot\dot{\vec{E}}_2-\dot{\vec{J}}_{cond,2}\cdot\dot{\vec{E}}_1\right)-\\&\left(j\omega\dot{\vec{E}}_1\cdot\dot{\vec{D}}_2-j\omega\dot{\vec{E}}_2\cdot\dot{\vec{D}}_1\right)+\left(j\omega\dot{\vec{H}}_1\cdot\dot{\vec{B}}_2-j\omega\dot{\vec{H}}_2\cdot\dot{\vec{B}}_1\right)\end{aligned} \qquad \text{(A.6)}$$

Due to the assumed symmetry of the magnetic permeability, dielectric permittivity and electric conductivity tensors, the following equalities are verified:

$$\dot{\vec{J}}_{cond,1}\cdot\dot{\vec{E}}_2=\dot{\vec{J}}_{cond,2}\cdot\dot{\vec{E}}_1 \qquad j\omega\dot{\vec{E}}_1\cdot\dot{\vec{D}}_2=j\omega\dot{\vec{E}}_2\cdot\dot{\vec{D}}_1 \qquad j\omega\dot{\vec{H}}_1\cdot\dot{\vec{B}}_2=j\omega\dot{\vec{H}}_2\cdot\dot{\vec{B}}_1$$

Then, equation (A.6) reduces to the following one:

$$\nabla\cdot\left(\dot{\vec{E}}_1\times\dot{\vec{H}}_2-\dot{\vec{E}}_2\times\dot{\vec{H}}_1\right)=\dot{\vec{J}}^{\,i}_1\cdot\dot{\vec{E}}_2-\dot{\vec{J}}^{\,i}_2\cdot\dot{\vec{E}}_1 \qquad \text{(A.7)}$$

This is the **local version** of Lorentz reciprocity theorem for linear, nonhomogeneous, anisotropic, and lossy dispersive medium.

From (A.7) it is possible to obtain an integral version of Lorentz reciprocity theorem, that will be used to discuss the derivation of formula (1) and to derive formula (2)

To do this, let as consider a bounded orientable and smooth region $R_s\subset R^3$. Applying Gauss' theorem to the left-hand member of equation (A.7) we obtain, being $\vec{n}$ the outward normal to the boundary $\partial R_s$ of the region $R_s$, the following equality follows:

$$\int_{R_s}\nabla\cdot\left(\dot{\vec{E}}_1\times\dot{\vec{H}}_2-\dot{\vec{E}}_2\times\dot{\vec{H}}_1\right)dV=\int_{\partial R_s}\vec{n}\cdot\left(\dot{\vec{E}}_1\times\dot{\vec{H}}_2-\dot{\vec{E}}_2\times\dot{\vec{H}}_1\right)dS \qquad \text{(A.8)}$$

From (A.7) and (A.8):

$$\int_{\partial R_s}\vec{n}\cdot\left(\dot{\vec{E}}_1\times\dot{\vec{H}}_2-\dot{\vec{E}}_2\times\dot{\vec{H}}_1\right)dS=\int_{R_s}\dot{\vec{J}}^{\,i}_1\cdot\dot{\vec{E}}_2\,dV-\int_{R_s}\dot{\vec{J}}^{\,i}_2\cdot\dot{\vec{E}}_1\,dV \qquad \text{(A.9)}$$

This is an **integral version** of Lorentz reciprocity theorem for linear, nonhomogeneous, anisotropic, and lossy dispersive medium.

Suppose now that the vector phasor current field $\dot{\vec{J}}^{\,i}_1(\omega;\vec{r})$ is zero outside a region $R_1\subset R_s$ and the vector phasor current field $\dot{\vec{J}}^{\,i}_2(\omega;\vec{r})$ is zero outside a region $R_2\subset R_s$. Then we have the equalities:

$$\int_{R_s}\dot{\vec{J}}^{\,i}_1(\omega;\vec{r})\cdot\dot{\vec{E}}_2(\omega;\vec{r})dV=\int_{R_1}\dot{\vec{J}}^{\,i}_1(\omega;\vec{r})\cdot\dot{\vec{E}}_2(\omega;\vec{r})dV \qquad \text{(A.10.1)}$$

$$\int_{R_s}\dot{\vec{J}}^{\,i}_2(\omega;\vec{r})\cdot\dot{\vec{E}}_1(\omega;\vec{r})dV=\int_{R_2}\dot{\vec{J}}^{\,i}_2(\omega;\vec{r})\cdot\dot{\vec{E}}_1(\omega;\vec{r})dV \qquad \text{(A.10.2)}$$

If we consider that $R_s$ grows extending to all space $R^3$ and we suppose that the surface integral $\int_{\partial R_s} \vec{n}\cdot\left(\dot{\vec{E}}_1(\omega;\vec{r})\times\dot{\vec{H}}_2(\omega;\vec{r})-\dot{\vec{E}}_2(\omega;\vec{r})\times\dot{\vec{H}}_1(\omega;\vec{r})\right)dS$ tends to zero while $R_s$ grows, from equations (A.9), (A.10.1) and (A.10.2) we obtain the following version of Lorentz reciprocity theorem (sometimes called Lorentz-Carson reciprocity):

$$\int_{R_1} \dot{\vec{J}}_1^{i}(\omega;\vec{r})\cdot\dot{\vec{E}}_2(\omega;\vec{r})\,dV = \int_{R_2} \dot{\vec{J}}_2^{i}(\omega;\vec{r})\cdot\dot{\vec{E}}_1(\omega;\vec{r})\,dV \qquad \text{(A.11)}$$

The cancellation of the surface integral when the integration embraces the whole space can be justified for radiative fields applying various arguments (Kong, 1986; Rothwell and Cloud, 2001).

When the medium is lossless at infinity, assuming that this medium is homogeneous and isotropic far away from the sources, the asymptotic behavior of the radiated fields produced by the spatially bounded distribution of impressed electric current sources makes $\dot{\vec{E}}_1(\omega;\vec{r})\times\dot{\vec{H}}_2(\omega;\vec{r})-\dot{\vec{E}}_2(\omega;\vec{r})\times\dot{\vec{H}}_1(\omega;\vec{r})$ tend to zero fast enough, considering that the other fields, nonradiated, always tend to zero faster than the radiated fields.
Another justification is obtained working with a sequence of progressively less lossy media, because if a medium is lossy (provided the losses are different from zero) the surface integral vanishes. In the limit we can expect the surface integral to be zero for a lossless medium.

For non-radiative fields, like the quasi-stationary ones, when distances $r$ from the sources increases, the fields tend to zero like the static fields do. As consequence, each of the fields in $\dot{\vec{E}}_1(\omega;\vec{r})\times\dot{\vec{H}}_2(\omega;\vec{r})-\dot{\vec{E}}_2(\omega;\vec{r})\times\dot{\vec{H}}_1(\omega;\vec{r})$ tends to zero as $\frac{1}{r^2}$ so, when the surface recedes to infinity, the limit of the surface integral is zero.

**Appendix B: Quasi-stationary conditions**

Under quasi-stationary conditions each field oscillates in phase at different points in the material medium. As shown by Plonsey and Heppner (Plonsey and Heppner, 1967), for a linear, homogeneous, and isotropic volume conductor, quasi-stationary conditions may be assumed if propagation effects, capacitive effects and inductive effects produced by bioelectric sources distributed in the volume conductor are neglectable, and the normal component of the electric field in the boundary of the volume conductor can be set equal to zero.
In this appendix we extend their results to assure quasi-stationary conditions for a linear, inhomogeneous, and anisotropic volume conductor.

Here we emphasize the magneto-quasi-stationary (MQS) approximation, which allowed us to deduce the formula (22) for the magnetic vector lead field induced in the tissues by the energized coil.
In addition to allowing the foundation of the magnetic lead field, the MQS approximation can be applied to mathematical models that describe the magnetic stimulation of biological tissues. In both cases the magnetic measurement or the magnetic stimulation is done by non-contact devices.

There is another approximation, the electro-quasi-stationary (EQS) one, that neglects $\frac{\partial \vec{B}}{\partial t}$ in Faraday's induction law (equation (A.1.2) and assumes the validity of static relation $\nabla \times \vec{E} = \vec{0}$. The EQS approximation neglects inductive effects in a volume conductor. As consequence, this approximation neglects the magnetic energy relative to the electric energy (Haus and Melcher, 1989).
So, in the EQS approximation the electric field derives from a scalar potential:

$$\vec{E}(t,\vec{r}) = -\nabla\varphi(t,\vec{r})$$

If the volume conductor is linear, inhomogeneous, isotropic, and with a nondispersive dielectric permittivity $\varepsilon(\vec{r})$ , the relation between the electric displacement field and the electric field is given by $\vec{D}(t,\vec{r}) = \varepsilon(\vec{r})\vec{E}(t,\vec{r})$ while $\vec{D}$ verifies equation (A.1.4).
For low frequencies the dispersive effects in the electric conductivity can be neglected (Foster, 2000), so the density of electric conduction current is given as a function of the electric vector field by the equation $\vec{J}_{cond}(t,\vec{r}) = \sigma(\vec{r})\vec{E}(t,\vec{r})$
From the equation of conservation of charge (A.1.7), from (A.1.4), from the constitutive relation $\vec{D} = \varepsilon\vec{E}$ and from $\vec{E} = -\nabla\varphi$ , it follows that the scalar potential $\varphi$ verifies:

$$\nabla\cdot\left(\sigma(\vec{r})\nabla\varphi(t,\vec{r}) + \varepsilon(\vec{r})\frac{\partial}{\partial t}\nabla\varphi(t,\vec{r})\right) = \nabla\cdot J^i(t,\vec{r})$$

The conductivity $\sigma(\vec{r})$ and dielectric permittivity $\varepsilon(\vec{r})$ fields as well as the source term $\nabla\cdot J^i(t,\vec{r})$ are assumed known. Here $J^i(t,\vec{r})$ is the field of impressed currents in the volume conductor, due to the double layers of electric currents at the level of the cell membranes, that are part of certain physiological or pathophysiological processes in biological tissues. In the boundary between the tissues and its environment, the normal component of the combined electric current density field (conduction plus displacement currents) is set equal to zero.
Once the electric field has been determined, both the curl and the divergence of the magnetic field are determined, so the magnetic field in the volume conductor and in its environment can be calculated also (Haus and Melcher, 1989).
If the displacement currents in the volume conductor can be neglected, and a conductivity tensor field $\tilde{\sigma}(\vec{r})$ allows us to describe inhomogeneity and anisotropy, the scalar potential verifies $\nabla\cdot\left(\tilde{\sigma}(\vec{r})\cdot\nabla\varphi(t,\vec{r})\right) = \nabla\cdot J^i(t,\vec{r})$ in the volume conductor, and the normal component of the conduction current vanishes at its boundary. This restricted version of the EQS approximation is often applied to the mathematical analysis of both, the measurement of physiological or pathophysiological electric fields (electrocardiography (ECG), electroencephalography (EEG), electromyography (EMG) and the electric stimulation of biological tissues, including cardiac and nervous pacing. In both cases, measurement of voltages or stimulation of tissues, is done by contact electrodes.
The voltage that appears between a pair of measuring electrodes can be related with the field of impressed currents (due to physiological or pathophysiological processes) and an electric lead vector field, in the framework of the theorem of reciprocity, for a heterogeneous and anisotropic conductive medium.

As the EQS approach can be applied to study the electric and magnetic fields originated in physiological or pathophysiological processes, or to the electric currents in the tissues produced by external electric fields or voltages, it is applicable to the description of the currents generated in the volume conductor of the tissues. Because of this, the EQS approximation affords a suitable framework in the interpretation of experimental results, in clinical diagnosis and in the design of equipment able to measure or produce electric fields and currents in the human body.

This restricted version is the most useful to calculate the electric and magnetic fields produced by the field of impressed currents $\vec{J}_T^{\,i}(t,\vec{r})$ that appear in the original Plonsey´s formula (1) and in the modified formulas (2.1) and (2.2).

However, EQS approximation is not adequate to discuss the meaning of these formulas, once $\vec{J}_T^{\,i}(t,\vec{r})$ is supposed to be known. So, let us return to the discussion of the MQS approximation.

The MQS approximation neglects capacitive effects in a volume conductor, thus neglecting the electric energy relative to the magnetic energy (Haus and Melcher, 1989).

From Maxwell equations (A.2) in phasor vector form, and the constitutive relations (A.3), considering that the magnetic permeability is constant, assuming that when the volume conductor is in an external magnetic field, there are no impressed current sources due to physiological activity in the tissues, we have the following set of electromagnetic equations for the volume conductor of the biological tissues:

$$\nabla \times \dot{\vec{H}}_{Tc}(\omega;\vec{r}) = \left[\tilde{\sigma}(\omega;\vec{r}) + j\omega\tilde{\varepsilon}(\omega;\vec{r})\right] \cdot \dot{\vec{E}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.1.1)}$$

$$\nabla \times \dot{\vec{E}}_{Tc}(\omega;\vec{r}) = -j\omega\mu \cdot \dot{\vec{H}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.1.2)}$$

$$\nabla \cdot \dot{\vec{H}}_{Tc}(\omega;\vec{r}) = 0 \qquad \text{(B.1.3)}$$

$$\nabla \cdot \left[\tilde{\varepsilon}(\omega;\vec{r}) \cdot \dot{\vec{E}}_{Tc}(\omega;\vec{r})\right] = \dot{\rho}_{free}(\omega;\vec{r}) \qquad \text{(B.1.4)}$$

Equations (B.1) can be applied to a linear, inhomogeneous, anisotropic, and dispersive volume conductor filling a region $R_T$. The magnetic permeability $\mu$ of the medium as well as the magnetic permeability of the environment adjacent to the medium can be assumed equal to $\mu_0$.

At the boundary $\partial R_T$ between the volume conductor and the environment the conservation of charge implies that the normal component of the total phasor vector current density (conduction plus displacement) must be continuous through the boundary.

If $\vec{n}$ is the normal vector at each point of the boundary, and when the environment has zero conductivity and its dielectric permittivity is near the vacuum value the normal components of the total current are equal at the boundary:

$$\vec{n} \cdot \left[\tilde{\sigma}(\omega;\vec{r}) + j\omega\tilde{\varepsilon}(\omega;\vec{r})\right] \cdot \dot{\vec{E}}_{Tc}(\omega;\vec{r}) = j\omega\varepsilon_0 \vec{n} \cdot \dot{\vec{E}}_{env}(\omega;\vec{r}) \qquad \vec{r} \in \partial R_T \qquad \text{(B.1.5)}$$

Let us decompose the Fourier transforms of the conductivity and the dielectric permittivity tensors as follows:

$$\tilde{\sigma}(\omega;\vec{r}) = \sigma(\omega)\underline{\tilde{\sigma}}(\omega;\vec{r}) \qquad \text{(B.2.1)} \qquad\qquad \tilde{\varepsilon}(\omega;\vec{r}) = \varepsilon(\omega)\underline{\tilde{\varepsilon}}(\omega;\vec{r}) \qquad \text{(B.2.2)}$$

In equations (B.2) the positive scalar functions $\sigma(\omega)$ and $\varepsilon(\omega)=\varepsilon_0\varepsilon_r(\omega)$ are representative values of the conductivity and the dielectric permittivity (like the functions of frequency shown in Figure 2) so that the dimensionless components of the tensors $\underline{\tilde{\sigma}}(\omega;\vec{r})$ and $\underline{\tilde{\varepsilon}}(\omega;\vec{r})$ are of the numerical order of unity or less.

Now, considering (B.2.1), equation (B.1.1) can be rewritten:

$$\nabla\times\dot{\vec{H}}_{Tc}(\omega;\vec{r})=\sigma(\omega)\left[\underline{\tilde{\sigma}}(\omega;\vec{r})+j\left(\frac{\omega\varepsilon(\omega)}{\sigma(\omega)}\right)\underline{\tilde{\varepsilon}}(\omega;\vec{r})\right]\cdot\dot{\vec{E}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.3.1)}$$

The boundary condition (B.1.5) can be rewritten as follows:

$$\vec{n}\cdot\left[\underline{\tilde{\sigma}}(\omega;\vec{r})+j\left(\frac{\omega\underline{\tilde{\varepsilon}}(\omega;\vec{r})}{\sigma(\omega)}\right)\right]\cdot\dot{\vec{E}}_{Tc}(\omega;\vec{r})=j\left(\frac{\omega\varepsilon_0}{\sigma(\omega)}\right)\vec{n}\cdot\dot{\vec{E}}_{env}(\omega;\vec{r})\quad\vec{r}\in\partial R_T \qquad \text{(B.3.2)}$$

In the frequency interval between 0 and 10 kHz the dimensionless parameter $\frac{\omega\varepsilon_0}{\sigma(\omega)}\cong\frac{(6.28\times10^4)\times(8.85\times10^{-12})}{0.22}<10^{-6}$ so, as we can expect that the electric fields in one side and the other from the border do not differ too much, (B.3.2) can be approximated by:

$$\vec{n}\cdot\left[\tilde{\sigma}(\omega;\vec{r})+j\omega\tilde{\varepsilon}(\omega;\vec{r})\right]\cdot\dot{\vec{E}}_{Tc}(\omega;\vec{r})=0 \qquad \vec{r}\in\partial R_T \qquad \text{(B.3.3)}$$

The normal component of the total phasor vector current density (conduction plus displacement) in the volume conductor must vanish at the boundary:

$$\vec{n}\cdot\vec{J}_{total,Tc}(t,\vec{r})=0 \qquad \vec{r}\in\partial R_T \qquad \text{(B.3.4)}$$

If the dimensionless number $\frac{\omega\varepsilon(\omega)}{\sigma(\omega)}$ is at least of a numerical order of magnitude less than 1 in the considered interval of frequencies, capacitive effects in the bulk of the volume conductor can be neglected and (B.3.1) can be approximated by:

$$\nabla\times\dot{\vec{H}}_{Tc}(\omega;\vec{r})=\sigma(\omega)\underline{\tilde{\sigma}}(\omega;\vec{r})\cdot\dot{\vec{E}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.3.5)}$$

Then, the boundary condition (B.3.4) is reduced to equation (14), which is thus supported:

$$\vec{n}\cdot\vec{J}_{con,Tc}(t,\vec{r})=0 \qquad \vec{r}\in\partial R_T$$

Both, the conductivity, and the permittivity tensors are invertible operators, so (B.3.2) can be cast in the following equivalent form:

$$\left[\underline{\tilde{\sigma}}(\omega;\vec{r})\right]^{-1}\cdot\nabla\times\dot{\vec{H}}_{Tc}(\omega;\vec{r})=\sigma(\omega)\dot{\vec{E}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.3.6)}$$

Then we have: $$\nabla\times\left(\left[\underline{\tilde{\sigma}}(\omega;\vec{r})\right]^{-1}\cdot\nabla\times\dot{\vec{H}}_{Tc}(\omega;\vec{r})\right)=\sigma(\omega)\nabla\times\dot{\vec{E}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.4.1)}$$

From (B.4.1) and (B.1.2):

$$\nabla\times\left(\left[\underline{\tilde{\sigma}}(\omega;\vec{r})\right]^{-1}\cdot\nabla\times\dot{\vec{H}}_{Tc}(\omega;\vec{r})\right)=-j\omega\mu\sigma(\omega)\dot{\vec{H}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.4.2)}$$

Let us introduce the following definitions:

$$\omega\mu\sigma(\omega)=\frac{1}{\delta^2(\omega)} \text{ (B.5.1)} \quad \underline{\vec{r}}=\frac{1}{\ell}\vec{r} \text{ (B.5.2)} \quad \nabla=\frac{1}{\ell}\underline{\nabla} \text{ (B.5.3)} \quad \in=\left(\frac{\ell}{\delta(\omega)}\right)^2 \text{ (B.5.4)}$$

These are, respectively a skin depth $\delta(\omega)$ (Landau and Lifshitz, 1960), a dimensionless position vector $\underline{\vec{r}}$ , a dimensionless gradient operator $\underline{\nabla}$ and a dimensionless parameter $\in$, square of the quotient between a representative length $\ell$ of the volume conductor (this could be a maximum diameter of the body) and the skin depth for the considered angular frequency $\omega$.
From equations (B.5) and equation (B.4.2) we obtain:

$$\underline{\nabla}\times\left(\left[\underline{\tilde{\sigma}}(\omega;\vec{r})\right]^{-1}\cdot\underline{\nabla}\times\dot{\vec{H}}_{Tc}(\omega;\vec{r})\right)=-j\in\dot{\vec{H}}_{Tc}(\omega;\vec{r}) \qquad \text{(B.6)}$$

In the interval of frequencies between 0 Hz and 10 kHz the conductivity $\sigma(\omega)$ can be considered as constant (Foster, 2000).
A representative conductivity for a homogeneous and isotropic model of the volume conductor could be 0.22 S/m and an upper bound, if asymptotic analysis methods are going to be applied, could be 1.0 S/m (Foster, 2000; Gabriel and Peyman, 2019).
The oscillating external magnetic field penetrates in the volume conductor to a distance:

$$\delta(\omega)\approx\frac{1}{\sqrt{\mu\sigma\omega}} \qquad \text{(B.7)}$$

So, if propagation, capacitive, and inductive effects produced by bioelectric sources distributed in the volume conductor, can be neglected as is the case when frequencies belong to the interval between 0 and 10kHz (Plonsey and Heppner, 1967; Foster, 2000; Gabriel and Peyman, 2019) and if the volume conductor maximum size $\ell$ is small enough relative to $\delta(\omega)$ so that the dimensionless number $\in=\left(\frac{\ell}{\delta(\omega)}\right)^2\ll 1$, then the magnetic field spatial distribution at any instant will be the same as it would be in a steady state with the corresponding value of the current sources sustained in time.
This can be seen by a regular perturbation analysis in the spirit of the one to proposed, in terms of increasing powers of the angular frequency for a homogeneous, isotropic, and non-dispersive volume conductor, by Fano, Chu and Adler, (Fano, Chu and Adler, 1960).

A conservative estimation of $\in$ can be done taking $\omega=2000\pi$ Hz, $\mu=\mu_0=4\pi\times10^{-7}$ F/m and $\sigma=1$ S/m. Then $\in$ is less than 0.001.
A regular perturbation schema gives the solution of equation (B.6):

$$\dot{\vec{H}}_{Tc}=\dot{\vec{H}}_0+\in\dot{\vec{H}}_1+\in^2\dot{\vec{H}}_2+\ldots \qquad \text{(B.8)}$$

This can be decomposed in the addition of a primary $\dot{\vec{H}}_0=\dot{\vec{H}}_p$ (external) phasor vector amplitude of magnetic field produced by the coil and a secondary $\dot{\vec{H}}_s=\in\dot{\vec{H}}_1+\in^2\dot{\vec{H}}_2+\ldots$ phasor vector amplitude of the magnetic field related with the eddy currents produced in the volume conductor:

$$\dot{\vec{H}}_{Tc}=\dot{\vec{H}}_p+\dot{\vec{H}}_s \qquad \text{(B.9)}$$

The primary magnetic field verifies the stationary equation in the frequency domain:

$$\underline{\nabla}\times\left(\left[\underline{\tilde{\sigma}}(\omega;\vec{r})\right]^{-1}\cdot\underline{\nabla}\times\dot{\vec{H}}_p(\omega;\vec{r})\right)=\vec{0} \qquad \text{(B.10)}$$

Going back to the time domain and considering a frequency independent conductivity tensor a stationary equation for the primary field results. Its solution can be factorized:

$$\vec{H}_p(t,\vec{r}) = f_H(t)\vec{H}_{p,0}(\vec{r}) \qquad \text{(B.11)}$$

Besides $\vec{H}_p(t,\vec{r}) = \frac{1}{\mu}\nabla\times\vec{A}_p(t,\vec{r})$ So, the primary vector potential can be factorized:

$$\vec{A}_p(t,\vec{r}) = \vec{A}_{Tc}(t,\vec{r}) = f_H(t)\vec{\Lambda}_{Tc}(\vec{r}) \qquad \text{(B.12)}$$

The function $f_H(t)$ must be proportional to the electric current $i_c(t)$ in the energized coil.

Redefining (if necessary) $\vec{H}_{p,0}(\vec{r}) = \nabla\times\vec{\Lambda}_{Tc}(\vec{r})$ it is possible to write:

$$f_H(t) = i_c(t) \qquad \text{(B.13)}$$

Returning to the time domain in $\nabla\times\dot{\vec{E}}(\omega;\vec{r}) = -j\omega\mu\cdot\dot{\vec{H}}(\omega;\vec{r})$ (equation (B.1.2)) we have:

$$\nabla\times\vec{E}_{Tc}(t,\vec{r}) = -\mu\cdot\frac{\partial}{\partial t}\vec{H}_p(t,\vec{r}) = -\mu\cdot\left(\frac{d}{dt}f_H(t)\right)\vec{H}_{p0}(\vec{r}) \qquad \text{(B.14)}$$

From (B.13) and (B.14) it follows, defining $f_E(t) = \frac{d}{dt}i_c(t)$:

$$\vec{E}_{Tc}(t,\vec{r}) = f_E(t)\vec{\mathcal{E}}^m_{Tc}(\vec{r}) = \left(\frac{d}{dt}i_c(t)\right)\vec{\mathcal{E}}^m_{Tc}(\vec{r}) \qquad \text{(B.15)}$$

However, from (A.1.6): $$\vec{E}_{Tc}(t,\vec{r}) = -\nabla\varphi_{Tc}(t,\vec{r}) - \frac{\partial}{\partial t}\vec{A}_{Tc}(t,\vec{r}) \qquad \text{(B.16)}$$

Then, from (B.12) and (B.16):

$$\varphi_{Tc}(t,\vec{r}) = f_E(t)\Phi_{Tc}(\vec{r}) = \left(\frac{d}{dt}i_c(t)\right)\Phi_{Tc}(\vec{r}) \qquad \text{(B.17)}$$

Substituting the current $i_c(t)$ imposed to the coil by the complex harmonic component $\hat{i}_c(t) = e^{j\omega t}I_0(\omega)$ , from (B.15) we have the following formula for the phasor electric field necessary to the derivation of the modified formula:

$$\dot{\vec{E}}(\omega,\vec{r}) = j\omega I_0(\omega)\vec{\mathcal{E}}^m_{Tc}(\vec{r}) \qquad \text{(B.18)}$$

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